Using Rotational Matrices to Generate Triplex Algebras

The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative.

—John Baez

To calculate three-dimensional fractals we need a triplex algebra, ie an arithmetic for three-dimensional co-ordinates, to allow us to add and multiply numbers such as (1, 2, 6). In the spirit of this quote, it seems that any triplex algebra belongs to a degenerate branch of the family tree. Be that as it may, the beauty and complexity of the fractal images it generates justify its existence.

This paper contains no original ideas. It merely attempts to systematise the mathematics introduced by Daniel White and Paul Nylander in pursuit of the three-dimensional equivalent of the Mandelbrot set. This much sought mythical beast has been variously called the Mandelbulb, Mandalabrot, and even Manhanabrot.

Rationale

Multiplying a complex number by another complex number is equivalent to stretching the first number by the modulus of the second, and rotating it by the angle of the second number. To see this, consider two complex numbers in polar form, $re^{i\theta}$ and $se^{i\phi}$. Their product is $rse^{i(\theta+\phi)}$, so the resulting number has modulus equal to the product of the moduli of the two original numbers, and its angle is the sum of their angles. This idea can be extended to triplex numbers, but with two caveats, (1) the polar form is not unique, and (2) the triplex numbers do not form a well-behaved algebra.

There are three basic rotation matrices in three dimensions, corresponding to rotations around the x, y and z axes:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For instance, $R_x(\theta)$ rotates the z-axis towards the y-axis by the angle θ, so that the point (x, y, z) becomes (x, ycosθ - zsinθ, zcosθ + ysinθ). This result comes from the matrix multiplication of $R_x(\theta)$ by the vector (x, y, z) interpreted as a matrix with one vertical column. (See http://en.wikipedia.org/wiki/Matrix_multiplication.)

Taking all pairwise products of the rotational matrices, we obtain six matrix terms: $R_x(\theta)R_y(\phi)$, $R_y(\theta)R_x(\phi)$, $R_z(\theta)R_y(\phi)$, $R_y(\theta)R_z(\phi)$, $R_z(\theta)R_x(\phi)$, and $R_x(\theta)R_z(\phi)$.

Each of the above six matrix products can be interpreted as a rotation through two angles. Bearing in mind the parallel with complex numbers, this is analogous to multiplying two triplex numbers. To obtain the actual formulas for triplex multiplication we multiply the matrix product by the vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1). The three resulting triplex values are the three columns of the matrix.

In terms of producing graphics, only two of these triplex values are of interest in each matrix product. The ones that have zero terms are degenerate from a graphical perspective.

Having obtained 12 triplex polar forms we now try all combinations of plus and minus for θ and ϕ. This gives us 48 different formulas for the triplex polar form.

NB Maybe we should also interchange θ and ϕ, as these are defined differently. That might increase the possibilities to 96.

Each variation is effectively a formula for triplex polar form, similar to the familiar (cosθcosϕ, sinθcosϕ, sinϕ) generously popularised by Paul Nylander. Any complex number can be raised to a real power p using $(re^{i\theta})^p = r^p e^{pi\theta}$. By analogy with the complex formula, we define a triplex exponentiation formula along these lines:
$(x, y, z)^p = r^p(\cos(p\theta)\cos(p\phi), \sin(p\theta)\cos(p\phi), \sin(p\phi))$
where $r = \sqrt{(x^2 + y^2 + z^2)}$, $\theta = \text{atan2}(y, x)$ and $\phi = \arcsin(z/r)$. (1)

Note that p can be any real value, allowing us to define negative and fractional powers, in addition to using natural numbers.

The choice of a particular polar form also bestows a multiplication formula similar to:
$(x_1, y_1, z_1)*(x_2, y_2, z_2) = r_1r_2(\cos(\theta_1+\theta_2)\cos(\phi_1+\phi_2), \sin(\theta_1+\theta_2)\cos(\phi_1+\phi_2), \sin(\phi_1+\phi_2))$
where $r_1 = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$, $\theta_1 = \text{atan2}(y_1, x_1)$ and $\phi_1 = \arcsin(z_1/r_1)$ and
$r_2 = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$, $\theta_2 = \text{atan2}(y_2, x_2)$ and $\phi_2 = \arcsin(z_2/r_2)$. (2)

Note that a particular polar form, ie (cosθcosϕ, sinθcosϕ, sinϕ), has been used in the formulas above purely for ease of illustration. Each variant of the polar form generates a different pair of formulas, analogously to equations 1 and 2. The particular polar form used here occurs as #43 below.

Once triplex multiplication and exponentiation are both defined, it is feasible to calculate various fractal formulas and so to plot them as mandelbulbs.

The 48 combinations of cos and sin terms listed below each define a particular exponentiation and a corresponding multiplication formula. Each of these pairs of formulas can be used to graphically express a given fractal formula in a way that is different from that of other polar forms. The Visions of Chaos program, available from http://softology.com.au/ uses all 48 variations.

The 48 triplex polar forms are worked out below.

$$\mathbf{R_x(\theta)\,R_y(\phi)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix} * \begin{vmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{vmatrix}$$

Taking columns one and three, this gives us:
1) XpYp1 = (cosφ, sinθsinφ, -cosθsinφ) and
2) XpYp3 = (sinφ, -sinθcosφ, cosθcosφ). By swapping signs in the first term we get:
3) XpYn1 = (cosφ, -sinθsinφ, cosθsinφ), (negative φ)
4) XnYp1 = (cosφ, -sinθsinφ, -cosθsinφ), (negative θ)
5) XnYn1 = (cosφ, sinθsinφ, cosθsinφ) (negative φ & θ) and from the second term:
6) XpYn3 = (-sinφ, -sinθcosφ, cosθcosφ), (negative φ)
7) XnYp3 = (sinφ, sinθcosφ, cosθcosφ), (negative θ)
8) XnYn3 = (-sinφ, sinθcosφ, cosθcosφ) (negative φ & θ).

$$\mathbf{R_y(\theta)\,R_x(\phi)} = \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} * \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{vmatrix}$$

Taking columns two and three, this gives us:
9) YpXp2 = (sinθsinφ, cosφ, cosθsinφ) and
10) YpXp3 = (sinθcosφ, -sinφ, cosθcosφ). Signs swaps in the first term give:
11) YpXn2 = (-sinθsinφ, cosφ, -cosθsinφ), (negative φ)
12) YnXp2 = (-sinθsinφ, cosφ, cosθsinφ), (negative θ)
13) YnXn2 = (sinθsinφ, cosφ, -cosθsinφ) (negative φ & θ) and from the second term:
14) YpXn3 = (sinθcosφ, sinφ, cosθcosφ), (negative φ)
15) YnXp3 = (-sinθcosφ, -sinφ, cosθcosφ), (negative θ)
16) YnXn3 = (-sinθcosφ, sinφ, cosθcosφ) (negative φ & θ).

$$R_z(\theta)R_x(\phi) = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -\sin\theta\cos\phi & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{vmatrix}$$

Taking columns two and three, this gives us:
17) ZpXp2 = (-sinθcosϕ, cosθcosϕ, sinϕ) and
18) ZpXp3 = (sinθsinϕ, -cosθsinϕ, cosϕ). By swapping signs in the first term we get:
19) ZpXn2 = (-sinθcosϕ, cosθcosϕ, -sinϕ), (negative ϕ)
20) ZnXp2 = (sinθcosϕ, cosθcosϕ, sinϕ), (negative θ)
21) ZnXn2 = (sinθcosϕ, cosθcosϕ, -sinϕ) (negative ϕ & θ) and from the second term:
22) ZpXn3 = (-sinθsinϕ, cosθsinϕ, cosϕ), (negative ϕ)
23) ZnXp3 = (-sinθsinϕ, -cosθsinϕ, cosϕ), (negative θ)
24) ZnXn3 = (sinθsinϕ, cosθsinϕ, cosϕ) (negative ϕ & θ).

$$R_x(\theta)R_z(\phi) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix} * \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{vmatrix}$$

Taking columns one and two, this gives us:
25) XpZp1 = (cosϕ, cosθsinϕ, sinθsinϕ) and
26) XpZp2 = (-sinϕ, cosθcosϕ, sinθcosϕ). By swapping signs in the first term we get:
27) XpZn1 = (cosϕ, -cosθsinϕ, -sinθsinϕ), (negative ϕ)
28) XnZp1 = (cosϕ, cosθsinϕ, -sinθsinϕ), (negative θ)
29) XnZn1 = (cosϕ, -cosθsinϕ, sinθsinϕ), (negative ϕ & θ) and from the 2nd term:
30) XpZn2 = (sinϕ, cosθcosϕ, sinθcosϕ), (negative ϕ)
31) XnZp2 = (-sinϕ, cosθcosϕ, -sinθcosϕ), (negative θ)
32) XnZn2 = (sinϕ, cosθcosϕ, -sinθcosϕ) (negative ϕ & θ).

$$\mathbf{R_y(\theta)R_z(\phi)} = \begin{vmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} * \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\cos\phi & -\cos\theta\sin\phi & \sin\theta \\ \sin\phi & \cos\phi & 0 \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

Taking columns one and two, this gives us:
33) YpZp1 = (cosθcosϕ, sinϕ, -sinθcosϕ) and
34) YpZp2 = (-cosθsinϕ, cosϕ, sinθsinϕ). By swapping signs in the first term we get:
35) YpZn1 = (cosθcosϕ, -sinϕ, -sinθcosϕ), (negative ϕ)
36) YnZp1 = (cosθcosϕ, sinϕ, sinθcosϕ), (negative θ)
37) YnZn1 = (cosθcosϕ, -sinϕ, sinθcosϕ), (negative ϕ & θ) and from the 2nd term:
38) YpZn2 = (cosθsinϕ, cosϕ, -sinθsinϕ), (negative ϕ)
39) YnZp2 = (-cosθsinϕ, cosϕ, -sinθsinϕ), (negative θ)
40) YnZn2 = (cosθsinϕ, cosϕ, sinθsinϕ) (negative ϕ & θ).

$$\mathbf{R_z(\theta)R_y(\phi)} = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} * \begin{vmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \cos\theta & \sin\theta\sin\phi \\ -\sin\phi & 0 & \cos\phi \end{vmatrix}$$

Taking columns one and three, this gives us:
41) ZpYp1 = (cosθcosϕ, sinθcosϕ, -sinϕ) and
42) ZpYp3 = (cosθsinϕ, sinθsinϕ, cosϕ). By swapping signs in the first term we get:
43) ZpYn1 = (cosθcosϕ, sinθcosϕ, sinϕ), (negative ϕ)
44) ZnYp1 = (cosθcosϕ, -sinθcosϕ, -sinϕ), (negative θ)
45) ZnYn1 = (cosθcosϕ, -sinθcosϕ, sinϕ), (negative ϕ & θ) and from the 2nd term:
46) ZpYn3 = (-cosθsinϕ, -sinθsinϕ, cosϕ), (negative ϕ)
47) ZnYp3 = (cosθsinϕ, -sinθsinϕ, cosϕ), (negative θ)
48) ZnYn3 = (-cosθsinϕ, sinθsinϕ, cosϕ) (negative ϕ & θ).

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